Special Case of 5-Body Central Configuration

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Abstract

This study examines central configurations within the framework of the 5-body problem, with a specific focus on systems where four of the masses lie on a common circle. This investigation does not assume that the four co-circular masses form a central configuration on their own. I explore potential placements of a fifth body that would transform the entire 5-body system into a central configuration. This approach broadens the scope of understanding by considering configurations that deviate from standard symmetry or balance.

Furthermore, the study delves into specific geometric arrangements of the co-circular four bodies, identifying cases where their particular shapes or distributions inherently prevent the inclusion of a fifth body that could satisfy the central configuration conditions. By combining analytical methods with geometric considerations, this work aims to provide deeper insights into the constraints and possibilities of central configurations in multi-body gravitational systems. These findings contribute to the broader understanding of celestial mechanics and the intricate interplay of forces in systems with nontrivial geometric constraints.

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Contents

1 Introduction

Central configurations in celestial mechanics are critical in understanding the dynamics of multi-body gravitational systems. These configurations correspond to special arrangements of masses where the mutual gravitational forces and the centrifugal forces, in a co-rotating reference frame, are in equilibrium. Such configurations play a pivotal role in studying the evolution, stability, and periodic solutions of multi-body systems.

In this section, we focus on analyzing central configurations involving five bodies, where four masses are arranged symmetrically, and a fifth mass is introduced to maintain or alter the central configuration. We begin with a simple symmetric case where four identical masses form a square, exploring the conditions under which a fifth mass can be added to preserve the central configuration. We then relax the symmetry to investigate scenarios where the four masses form two identical pairs or other geometric arrangements, such as rectangles, extending the discussion to cases where the configuration is no longer square but still satisfies the equilibrium conditions.

The analytical methods used involve detailed calculations of the system's center of mass, gravitational forces, and the resulting accelerations on each body. By leveraging symmetry and simplifying assumptions, we derive conditions under which the system maintains a central configuration and some impossible central configurations, offering insights into the broader study of gravitational systems with arbitrary mass distributions.

1.1 Central Configuration in the N-Body Problem

In celestial mechanics, the motion of particles in an N-body system is governed by the following equation of motion:

$$\ddot{x}_k = \sum_{i \neq k} \frac{m_i(x_i - x_k)}{|x_i - x_k|^3} \ , \tag{1.1}$$

where $x_k \in \mathbb{R}^2$ represents the position of the k-th particle, and $m_i > 0$ is its mass ($i = 1, \ldots, N$). The configuration of the system is defined as the vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^{2N}$, encapsulating the positions of all N particles.

The center of mass of the system is given by:

$$x_c = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \in \mathbb{R}^2 \ , \tag{1.2}$$

which represents the weighted average position of all the particles.

A collision configuration occurs when two or more particles occupy the same position in space, mathematically defined as:

$$\Delta = \{x \in \mathbb{R}^{3N} \mid x_i = x_j \text{ for some } i \neq j\} \ , \tag{1.3}$$

which represents all configurations in which at least one pair of particles has collided.

A central configuration is a special type of non-collision configuration $x \in \mathbb{R}^{2N} \backslash \Delta$ that satisfies a specific balance of forces. Formally, a configuration is a central configuration if there exists a constant $\lambda > 0$ such that for each $k = 1, \ldots, N$:

$$-\lambda(x_k - x_c) = \sum_{i \neq k} \frac{m_i(x_i - x_k)}{|x_i - x_k|^3} \ . \tag{1.4}$$

Physically, since the acceleration vector for the mass m_i is:

$$\mathscr{A}_i = \sum_{i \neq k} \frac{m_i(x_i - x_k)}{|x_i - x_k|^3} \ , \tag{1.5}$$

the central configuration is a state where the acceleration vector for each particle points to the center of mass and is proportional to its displacement from the center of mass, with the proportionality constant λ. This implies that when the particles are arranged in such a way initially the system remains dynamically balanced, maintaining a fixed shape during motion as Figure 1 illustrates. Central configurations often correspond to solutions where the system undergoes uniform rotation or a collapse to a singularity in a homothetic manner.

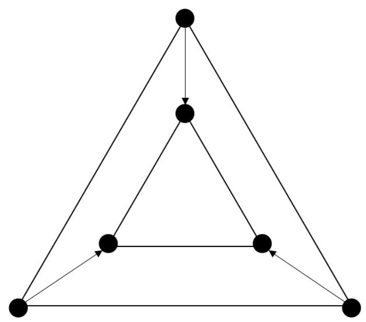

Figure 1: When three identical masses are released from a central configuration, specifically arranged in the shape of an equilateral triangle in the figure, the system will preserve its equilateral triangular shape as it moves.

Central configurations are also vital in the analysis of singularities. For example, in a homothetic collapse, the configuration might shrink or expand uniformly until the masses all collapse to a single point, or conversely, it could correspond to an expanding system.

In practical applications, central configurations provide insight into the qualitative behavior of N-body systems. They help predict long-term behavior, such as periodic motions and the possible configurations that particles might settle into in a gravitational system. These ideas are pivotal in celestial mechanics.

1.2 Central Configurations with Four Masses on the Same Circle

In the context of the five-body problem, where four masses are positioned on a circle, we aim to obtain a central configuration by placing the fifth body in the system. To do so, we need to analyze the forces acting on the system, considering both the gravitational interactions between the corner masses and the central mass.

Let the four masses m_1, m_2, m_3, m_4 be placed symmetrically on the circle of radius R, the positions of these four masses are:

$$x_i = R(\cos\theta_i, \sin\theta_i) \ , \quad i = 1, 2, 3, 4 \ . \tag{1.6}$$

We will focus on the case that the center of mass of these four masses lies on the origin, that is to say:

$$x_c = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4} = (0, 0) \ . \tag{1.7}$$

Then we try to place the fifth mass m_5 be placed in the system. The acceleration of m_5 due to any corner mass m_k, where $k = 1, 2, 3, 4$ is. In the five-body problem, where four masses are symmetrically arranged on a circle, our goal is to achieve a central configuration by introducing a fifth mass into the system. This involves analyzing the gravitational forces acting within the system, focusing on the interactions between the four peripheral masses and the central fifth mass.

In this report, we will explore special cases of five-body central configurations and examine the conditions under which a fifth mass can be added to a system of four masses. Specifically, we will investigate how different arrangements of the four masses, such as a square or circle, impact the possibility of achieving a central configuration. We will also explain why, in some cases, it is impossible to achieve a central configuration by adding a fifth mass when the four masses form certain shapes. Through this analysis, we aim to better understand the conditions that must be satisfied for such configurations to exist.

2 Four Bodies Form a Square

In this subsection, we focus on a symmetric arrangement of four identical masses m, positioned at the vertices of a square inscribed in a circle of radius R, centered at the origin. The goal is to determine where a fifth mass m_5 can be placed to satisfy the central configuration condition. Symmetry plays a crucial role in this analysis, as it significantly simplifies the calculations and provides insight into how equilibrium is achieved. By leveraging this symmetry, we investigate the conditions under which the fifth mass contributes to a stable central configuration and verify the forces acting on all masses. This examination provides a foundation for understanding more complex configurations involving identical and non-identical masses.

2.1 Four Identical Masses

Let's now consider an example where the four bodies of equal mass m form a square on a circle of radius R, centered at the origin. We aim to find where the fifth body m_5 can be placed to satisfy the central configuration condition.

Set four equal masses m are at the following positions:

$$x_1 = R(\cos 0, \sin 0) = (R, 0)\ , \quad x_2 = R(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (0, R)\ , \tag{2.1}$$

$$x_3 = R(\cos\pi, \sin\pi) = (-R, 0), \quad x_4 = R(\cos\frac{3\pi}{2}, \sin\frac{3\pi}{2}) = (0, -R)\ , \tag{2.2}$$

which forms a square on the circle as Figure 2 shows. Now we try to place the fifth body m_5 at an arbitrary position $x_5 = (a, b)$ in the plane. The center of mass of the system is:

$$\begin{aligned} x_c &= \frac{m(x_1 + x_2 + x_3 + x_4) + m_5 x_5}{4m + m_5} \\ &= \frac{m[(R,0) + (0,R) + (-R,0) + (0,-R)] + m_5 x_5}{4m + m_5} \\ &= \frac{m_5 x_5}{4m + m_5}\ . \end{aligned} \tag{2.3}$$

Now consider a symmetry, due to the symmetry, it makes sense to place the fifth body at the origin. Suppose $x_5 = (0, 0)$, that is, the fifth body is placed at the center of the square. In

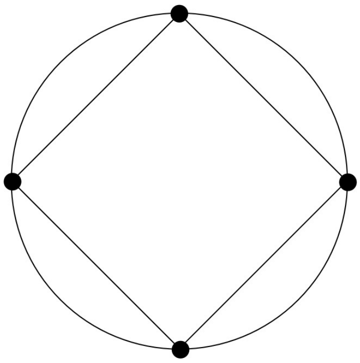

Figure 2: The positions of four masses on the same circle.

this case, the center of mass simplifies to:

$$x_c = \frac{m_5(0,0)}{4m + m_5} = (0,0)\ . \tag{2.4}$$

We want to check that placing m_5 at x_5 satisfies the condition for central configuration:

$$-\lambda(x_5 - x_c) = \sum_{i\neq 5} \frac{m_i(x_i - x_5)}{|x_i - x_5|^3}\ . \tag{2.5}$$

It is easy to see that the left-hand side $-\lambda(x_5 - x_c) = 0$, check that the right-hand side of the central configuration condition becomes zero as well:

$$\begin{aligned}
\mathscr{A}_5 &= \sum_{i\neq 5} \frac{m_i(x_i - x_5)}{|x_i - x_5|^3} \\
&= \frac{mx_1}{|x_1|^3} + \frac{mx_2}{|x_2|^3} + \frac{mx_3}{|x_3|^3} + \frac{mx_4}{|x_4|^3} \\
&= \frac{m}{R^3}\big[(R,0) + (0,R) + (-R,0) + (0,-R)\big] = (0,0)\ .
\end{aligned} \tag{2.6}$$

Thus, m_5 satisfies the central configuration condition.

Now we check the other masses $m_1,\ m_2,\ m_3,\ m_4$, by symmetry, we only need to check one of them, say m_1 here. The acceleration of m_1 due to m_2 is:

$$\begin{aligned}
\mathscr{A}_{2\to 1} &= \frac{m(0-R, R-0)}{\sqrt{R^2+R^2}^3} \\
&= \frac{m}{2\sqrt{2}R^2}(-1,1)\ .
\end{aligned} \tag{2.7}$$

Similarly, the acceleration of m_1 due to m_3 and m_4 are:

$$\begin{aligned}
\mathscr{A}_{3\to 1} &= \frac{m(-R-R, 0)}{(2R)^3} \\
&= \frac{m}{4R^2}(-1,0)\ ,
\end{aligned} \tag{2.8}$$

$$\mathscr{A}_{4\to 1} = \frac{m(0-R, -R-0)}{\sqrt{R^2+R^2}^3}$$

$$= \frac{m}{2\sqrt{2}R^2}(-1,-1) \; . \tag{2.9}$$

The the acceleration of m_1 due to m_5 is:

$$\mathscr{A}_{5\to1} = \frac{m_5(-R,0)}{R^3} = \frac{m_5}{R^2}(-1,0) \; . \tag{2.10}$$

Adding all forces on m_1:

$$\mathscr{A}_1 = \mathscr{A}_{2\to1} + \mathscr{A}_{3\to1} + \mathscr{A}_{4\to1} + \mathscr{A}_{5\to1} \; . \tag{2.11}$$

Perform the sum, we can obtain the net force:

$$\begin{aligned}\mathscr{A}_1 &= \frac{m}{2\sqrt{2}R^2}(-1,1) + \frac{m}{4R^2}(-1,0) + \frac{m}{2\sqrt{2}R^2}(-1,-1) + \frac{m_5}{R^2}(-1,0) \\ &= \left(\frac{m}{R^2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{4}\right) - \frac{m_5}{R^2}, 0\right) \; .\end{aligned} \tag{2.12}$$

For m_1 to satisfy the central configuration condition:

$$\mathscr{A}_1 = \left(\frac{m}{R^2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{4}\right), 0\right) = -\lambda(x_1 - x_c) = -\lambda(R,0) \; . \tag{2.13}$$

Equating components:

$$\frac{m}{R^2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{4}\right) = -\lambda R \iff \lambda = \frac{m}{R^3}\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) > 0 \; . \tag{2.14}$$

Hence, m_1 satisfies the central configuration condition.

By symmetry, the forces on m_2, m_3, and m_4 are identical in magnitude but rotated by $\pi/2$, π, and $3\pi/2$, respectively. Thus, the central configuration condition is satisfied for all four outer masses.

Thus, the fifth body m_5 placed at the origin satisfies the central configuration condition due to symmetry. The forces on m_1, m_2, m_3, and m_4 are consistent with the central configuration condition, and the system is balanced.

2.2 Two Identical Mass Pairs

Now we consider a relaxed version with a bit weaker symmetry. When the conditions are relaxed such that $m_1 = m_3$ and $m_2 = m_4$, the center of mass remains at the origin due to symmetry. We also place the fifth body m_5 on the center of mass of this system and prove this remains a central configuration.

The center of mass is given by:

$$x_c = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5}{m_1 + m_2 + m_3 + m_4 + m_5} \; . \tag{2.15}$$

Using $m_1 = m_3$ and $m_2 = m_4$, the center of mass becomes:

$$x_c = \frac{m_1(R,0) + m_2(0,R) + m_1(-R,0) + m_2(0,-R) + m_5(0,0)}{2m_1 + 2m_2 + m_5}$$

$$= \frac{(0,0)}{2m_1 + 2m_2 + m_5} = (0,0) \ . \tag{2.16}$$

Thus, the center of mass remains at the origin.

The force on m_5 due to the other four masses is:

$$\mathscr{A}_5 = \sum_{i=1}^{4} \frac{m_i(x_i - x_5)}{|x_i - x_5|^3} \ . \tag{2.17}$$

Place the fifth body on the center of mass; thus, $x_5 = x_c = (0,0)$, this simplifies to:

$$\mathscr{A}_5 = \sum_{i=1}^{4} \frac{m_i(x_i - x_5)}{|x_i - x_5|^3} = \sum_{i=1}^{4} \frac{m_i x_i}{|x_i|^3} \ . \tag{2.18}$$

The contributions from opposite pairs m_2, m_4 and m_1, m_3 are:

$$\begin{aligned} \mathscr{A}_5 &= \frac{m_1(R,0)}{R^3} + \frac{m_2(0,R)}{R^3} + \frac{m_3(-R,0)}{R^3} + \frac{m_4(0,-R)}{R^3} \\ &= \frac{m_1(R,0)}{R^3} + \frac{m_1(-R,0)}{R^3} + \frac{m_2(0,R)}{R^3} + \frac{m_2(0,-R)}{R^3} = (0,0) \ . \end{aligned} \tag{2.19}$$

Thus, the acceleration of m_5 is:

$$\mathscr{A}_5 = (0,0) \ . \tag{2.20}$$

Hence, m_5 satisfies the central configuration condition.

Now we check the masses m_1 and m_3, similarly, we only need to check one of them due to symmetry. For m_1 at $x_1 = (0,R)$, the net force is:

$$\mathscr{A}_1 = \sum_{i \neq 1} \frac{m_i(x_i - x_1)}{|x_i - x_1|^3} \ . \tag{2.21}$$

Substituting the positions of the other masses:

$$\mathscr{A}_1 = \frac{m_2(-R,-R)}{(R^2+R^2)^{3/2}} + \frac{m_3(-2R,0)}{(2R)^3} + \frac{m_4(-R,R)}{(R^2+R^2)^{3/2}} + \frac{m_5(-R,0)}{R^3} \ . \tag{2.22}$$

The $x-$ and $y-$ components of the force are:

$$\mathscr{A}_{1,x} = \frac{m_2(-R)}{(R^2+R^2)^{3/2}} + \frac{m_3(-2R)}{(2R)^3} + \frac{m_4(-R)}{(R^2+R^2)^{3/2}} + \frac{m_5(-R)}{R^3} \ , \tag{2.23}$$

$$\mathscr{A}_{1,y} = \frac{m_2(-R)}{(R^2+R^2)^{3/2}} + \frac{m_4 R}{(R^2+R^2)^{3/2}} = 0 \ . \tag{2.24}$$

The central configuration should satisfy:

$$\begin{aligned} \mathscr{A}_{1,x} &= \frac{-2m_2R}{(2R^2)^{3/2}} + \frac{-2m_3R}{(2R)^3} + \frac{-m_5R}{R^3} = -\lambda R \\ \iff \lambda &= \frac{2m_2}{(2R^2)^{3/2}} + \frac{2m_3}{(2R)^3} + \frac{m_5}{R^3} > 0 \ . \end{aligned} \tag{2.25}$$

Thus, m_1 and m_3 satisfy the condition of central configuration as well, for $\lambda = \frac{2m_2}{(2R^2)^{3/2}} + \frac{2m_3}{(2R)^3} + \frac{m_5}{R^3}$. However, by symmetry, the masses m_2 and m_4 satisfy the central configuration condition only when $\lambda = \frac{2m_3}{(2R^2)^{3/2}} + \frac{2m_2}{(2R)^3} + \frac{m_5}{R^3}$,so we can conclude that:

$$\lambda = \frac{2m_2}{(2R^2)^{3/2}} + \frac{2m_3}{(2R)^3} + \frac{m_5}{R^3} = \frac{2m_3}{(2R^2)^{3/2}} + \frac{2m_2}{(2R)^3} + \frac{m_5}{R^3} \ . \tag{2.26}$$

Since m_3 and m_2 are arbitrary, this configuration is a central configuration only when $m_1 = m_2 = m_3 = m_4$, which is equivalent to the case we studied in 2.1.

3 Four Bodies Not Form a Square

Now we want to find a central configuration that four bodies do not form a square. A natural guess is that when we add the fifth body into the system, the five-body system forms a pentagon. I will prove that this is a central configuration and thus it is possible to place the fifth body in the system to form a central configuration when four bodies do not form a square.

3.1 Four Bodies Form a Rectangle

We now consider the case where the four masses $m1,\ m2,\ m3,\ m4$ are arranged in a rectangle, as illustrated in Figure 3. Let four co-circular masses m_1, m_2, m_3, m_4, where $m_1 = m_3$ and $m_2 = m_4$, are at the corners of a rectangle, the positions are:

$$\begin{aligned} &x_1 = R(\cos\theta, \sin\theta) = (a, b) \ , \quad x_2 = R(\cos(\pi - \theta), \sin(\pi - \theta)) = (-a, b) \ , \\ &x_3 = R(\cos(\theta - \pi), \sin(\theta - \pi)) = (-a, -b) \ , \quad x_4 = R(\cos(-\theta), \sin(-\theta)) = (a, -b) \ . \end{aligned} \tag{3.1}$$

Place the fifth mass m_5 at the origin. The center of mass is given by:

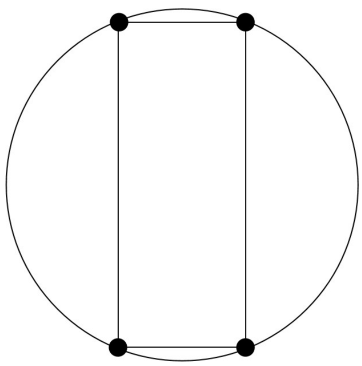

Figure 3: The case for four co-circular masses forming a rectangle configuration.

$$x_c = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5}{m_1 + m_2 + m_3 + m_4 + m_5} \ . \tag{3.2}$$

Using $m_1 = m_3$ and $m_2 = m_4$, the center of mass becomes:

$$x_c = \frac{m_1(a, b) + m_2(-a, b) + m_1(-a, -b) + m_2(a, -b) + m_5(0, 0)}{2m_1 + 2m_2 + m_5}$$

$$= \frac{(0,0)}{2m_1 + 2m_2 + m_5} = (0,0) \ . \tag{3.3}$$

Thus, the center of mass remains at the origin.

It is easy to check that m_5 satisfies the central configuration condition:

$$\begin{aligned} \mathscr{A}_5 &= \frac{m(a,b)}{R^3} + \frac{m(-a,b)}{R^3} + \frac{m(-a,-b)}{R^3} + \frac{m(a,-b)}{R^3} \\ &= \frac{m(a,b)}{R^3} + \frac{m(-a,-b)}{R^3} + \frac{m(-a,b)}{R^3} + \frac{m(a,-b)}{R^3} = (0,0) \ . \end{aligned} \tag{3.4}$$

For m_1 at $(-a,-b)$, the acceleration of m_1 is due to m_2, m_3, m_4, and m_5. The acceleration of m_1 due to the m_2 is:

$$\mathscr{A}_{2\to 1} = \frac{m(-2a,0)}{8a^3} = -\frac{m(1,0)}{4a^2} \ . \tag{3.5}$$

The acceleration of m_1 due to the m_3 is:

$$\mathscr{A}_{3\to 1} = \frac{m(-2a,-2b)}{\sqrt{4a^2+4b^2}^3} = -\frac{m(a,b)}{4\sqrt{a^2+b^2}^3} \ . \tag{3.6}$$

The acceleration of m_1 due to the m_4 is:

$$\mathscr{A}_{4\to 1} = \frac{m(0,-2b)}{8b^3} = -\frac{m(0,1)}{4b^2} \ . \tag{3.7}$$

The acceleration of m_1 due to the m_5 is:

$$\mathscr{A}_{5\to 1} = -\frac{m(a,b)}{\sqrt{a^2+b^2}^3} \ . \tag{3.8}$$

The total acceleration of m_1 is:

$$\begin{aligned} \mathscr{A}_1 &= \mathscr{A}_{2\to 1} + \mathscr{A}_{3\to 1} + \mathscr{A}_{4\to 1} + \mathscr{A}_{5\to 1} \\ &= -\left(\frac{5ma}{4R^3} + \frac{m}{4a^2}, \frac{5mb}{4R^3} + \frac{m}{4b^2}\right) \ . \end{aligned} \tag{3.9}$$

Equating both sides of the condition of the central configuration, we have

$$\lambda = \frac{5m}{4R^3} + \frac{m}{4a^3} = \frac{5m}{4R^3} + \frac{m}{4b^3} \ . \tag{3.10}$$

Therefore, this configuration is a central configuration only when $a = b$, which is equivalent to the case we studied in 2.1.

3.2 Four Bodies Forming a Kite

We now turn to the case where four co-circular masses, denoted m_1, m_2, m_3, m_4, form a configuration that resembles a kite, as shown in Figure 3.1. In this configuration, the masses are arranged such that the masses m_1 and m_3 are equal, and the masses m_2 and m_4 are also equal. However, we do not necessarily assume that these four masses lie on a common circle at this point.

Let the four masses m_1, m_2, m_3, m_4 be such that $m_1 = m_3$ and $m_2 = m_4$. Initially, we do not assume that these bodies lie on a circle. In this case, it is helpful to apply a theorem that describes the central configuration of such a system.

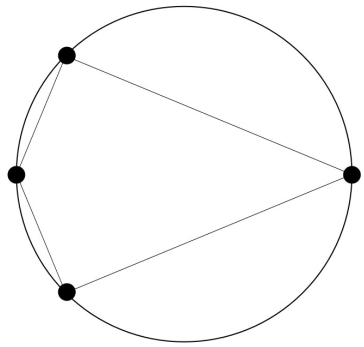

Figure 4: The case where four co-circular masses form a kite configuration.

Theorem 3.1. *Let four masses m_1, m_2, m_3, m_4, where $m_1 = m_3$ and $m_2 = m_4$ form a kite in the clockwise direction. Then we can obtain a central configuration by placing m_5 on the symmetry axis only when m_1, m_2, m_3, m_4 form a rhombus.*

In this setup, the configuration described by Theorem 3.1 requires that the four masses m_1, m_2, m_3, m_4 must form a rhombus in order for a central configuration to be achievable. This result is crucial because it implies that, for a central configuration to exist, there must be a certain geometric relationship between the masses that is stricter than simply arranging them in a kite shape. Specifically, the shape must be a rhombus, where the two diagonals are of equal length and the opposite angles are congruent.

Now, we proceed by placing the four masses m_1, m_2, m_3, m_4 on a common circle. This step is critical because, by doing so, we effectively impose the constraint that the masses are equidistant from the center of the circle. By Theorem 3.1, the central configuration can be achieved by placing m_5 along the symmetry axis of the system. However, as indicated by the theorem, this will only be possible if the arrangement of the masses m_1, m_2, m_3, m_4 satisfies the additional condition that they form a rhombus.

This result is equivalent to the case we explored earlier in Section 2.2. Therefore, we can conclude that when four masses are on the same circle and form a kite, we can not obtain the central configuration by placing the fifth body on the symmetry axis except the case we studied in 2.2.

3.3 Four Bodies Form a Trapezoid

We now consider the case where the four masses m_1, m_2, m_3, m_4 are arranged in a symmetric trapezoid, as illustrated in Figure 5.

In this configuration, the masses m_1 and m_3 are equal, as are the masses m_2 and m_4. However, we do not assume that these four masses are co-circular at this point.

Let the four masses m_1, m_2, m_3, m_4, where $m_1 = m_3$ and $m_2 = m_4$, form a symmetric trapezoid. To analyze the central configuration of this system, we utilize the following theorem.

Theorem 3.2. *Let four masses m_1, m_2, m_3, m_4, where $m_1 = m_3$ and $m_2 = m_4$ form a symmetric trapezoid in the clockwise direction. Then we can obtain a central configuration by placing m_5 on the intersection of diagonals only when m_1, m_2, m_3, m_4 form a rhombus.*

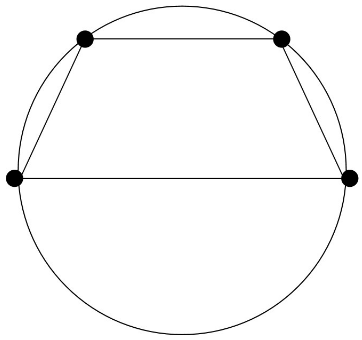

Figure 5: The case for four co-circular masses forming a symmetric trapezoid.

In this setup, the key condition for obtaining a central configuration is that the four masses m_1, m_2, m_3, m_4 must form a rhombus. The condition is stricter than simply forming a symmetric trapezoid, as it requires the trapezoid to have specific geometric properties: the opposite sides must be equal, and the diagonals must intersect at right angles. This specific form of the trapezoid ensures that a stable central configuration is achievable.

Now, we proceed by placing the four masses m_1, m_2, m_3, m_4 on a common circle. By doing this, we impose the constraint that all the masses are equidistant from the center of the circle. According to Theorem 3.2, we can achieve a central configuration by placing m_5 at the intersection of the diagonals, but only if the masses m_1, m_2, m_3, m_4 form a rhombus.

This result mirrors the analysis from Section 3.1. Therefore, we can conclude that when four masses are on the same circle and form a symmetric trapezoid, we can not obtain the central configuration by placing the fifth body on the intersection of diagonals except the case we studied in 3.1.

3.4 Co-Circular Five Identical Bodies

Let five masses $(x_1, x_2, x_3, x_4, x_5)$ be placed on the circumference of a circle and the system is symmetric, with the configuration centered at the origin of the circle (the center of mass coincides with the origin). The symmetry ensures equal distances between consecutive particles, forming a regular pentagon. The mutual distances r_{ij} between any two particles satisfy the geometry of a pentagon as Figure 6 shows. Mathematically, five particles lie on a circle

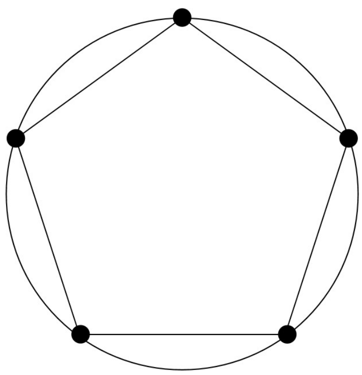

Figure 6: The co-circular five-body central configuration.

of radius r_C, with positions:

$$x_k = r_C(\cos\theta_k, \sin\theta_k), \quad \theta_k = \frac{2\pi k}{5}, \quad k = 0, 1, 2, 3, 4\ . \tag{3.11}$$

Using the geometry of the pentagon, the distance between two particles m_k and m_j is:

$$|x_k - x_j| = 2r_C \sin\left(\frac{\pi|k-j|}{5}\right)\ , \tag{3.12}$$

where $|k-j|$ is taken modulo 5. For a particle x_k, the gravitational force from a particle m_j is:

$$\mathscr{A}_{j\to k} = \frac{m(x_j - x_k)}{|x_j - x_k|^3}\ . \tag{3.13}$$

By symmetry, the net force contributions from all other particles will align radially toward the origin. In a regular pentagon, the particles are symmetrically distributed about the origin and the contributions of the forces from all other particles toward m_k must sum to a vector that points radially inward toward the center of the circle.

The total gravitational force acting on m_k is:

$$\mathscr{A}_k = \sum_{j\neq k} \frac{m(x_j - x_k)}{|x_j - x_k|^3}\ . \tag{3.14}$$

Due to the symmetry of the regular pentagon, the net direction of $\mathscr{A}_k$ is along $-x_k$, and its magnitude is proportional to $|x_k| = r_C$. Thus, we have:

$$\mathscr{A}_k = -\lambda x_k, \tag{3.15}$$

where $\lambda > 0$ is the proportionality constant, determined by the system's geometry and masses.

We verify the defining condition $-\lambda x_k = \mathscr{A}_k$ explicitly, since the force contributions from the nearest neighbors (adjacent particles) and the diagonal particles are symmetric, ensuring the net force aligns with $-x_k$ and the magnitude of the force, computed as:

$$\sum_{j\neq k} \frac{m}{|x_j - x_k|^2}\ , \tag{3.16}$$

which is proportional to $|x_k| = r_C$, satisfying the central configuration condition.

The symmetry of the regular pentagon ensures that the net gravitational force on each particle is radially directed toward the origin and the magnitude of this force is proportional to the displacement from the origin. Therefore, the regular pentagon satisfies the central configuration condition, proving it is a central configuration.

4 Conclusion

This study demonstrates that central configurations are fundamentally influenced by the symmetry and relative positioning of the masses. When four bodies of equal mass form a square, the placement of a fifth body at the center satisfies the central configuration condition due to symmetry. Similarly, when the symmetry is slightly relaxed, such as in cases where four masses form a rectangle or consist of two identical mass pairs, the configuration can still satisfy the central configuration condition under certain constraints. However, as symmetry decreases, the conditions required to maintain equilibrium become increasingly restrictive.

We also explored cases where four bodies do not form a square, such as pentagonal configurations, and verified that these systems can achieve central configurations with appropriately positioned masses. The results emphasize the critical role of geometric symmetry and mass distribution in ensuring the stability of central configurations. These findings enrich our understanding of equilibrium structures in gravitational systems and lay a foundation for further studies in more complex $N-$body arrangements.

References

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